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Symmetry analysis, conservation laws and exact solutions of ill-posed Boussinesq equation

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Abstract

This thesis presents a comprehensive study on the construction of conservation laws and the application of double reduction techniques to obtain exact solutions for the ill-posed boussinesq equation

$$u_{tt} = u_{xx} + u^2{}_{xx} + u_{xxxx}$$

Conservation laws are utilized to perform a double reduction of the original PDE. This reduction process involves two key stages: firstly, the application of the derived conservation laws to convert the PDE to ODE then reduction of the order of the ODE and secondly, the further simplification of the reduced equation by exploiting additional symmetries to obtain the exact solutions. The exact solutions are analyzed and graphically demonstrated to gain insight into the underlying physical and mathematical properties of the original PDE. The dissertation contributes to the field of applied mathematics by providing a rigorous framework for constructing conservation laws and applying reduction techniques to nonlinear PDEs. The exact solutions obtained not only advance the theoretical understanding of the equation but also offer potential applications in areas such as fluid dynamics, nonlinear optics, and other fields where similar equations arise.

Keywords: Boussineq Equation; Lie Point Symmetry; Conserved Vectors; Double Reductions; Exact Solutions

1. Introduction

Partial Differential Equations (PDEs) play a pivotal role in the modeling and analysis of various physical phenomena across disciplines such as physics, engineering, and applied mathematics [2,11,12]. These equations describe how physical quantities such as heat, sound, fluid flow, and electromagnetic fields change over space and time. The study of PDEs is fundamental to understanding the behavior of systems governed by the laws of nature, and their solutions provide insights into the dynamics of these systems.

Among the vast array of PDEs, nonlinear equations, in particular, pose significant challenges due to their complexity and the rich variety of behaviors they can exhibit. Nonlinear PDEs often describe processes where the effects of interactions cannot be simply added together, leading to phenomena such as shock waves, solitons, and turbulence [12]. Finding exact solutions to these equations is crucial, as they can serve as benchmarks for numerical simulations and offer deep insights into the underlying physical processes.

The Boussinesq equation, first derived by Joseph Boussinesq [1] is a nonlinear partial differential equation (PDE) that has far-reaching implications in various fields, including physics, engineering, and mathematics. Boussinesq[1], a French mathematician, presented his work in a paper titled "Théorie des ondes et des remous qui se propagent le long

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d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond" (Theory of waves and ripples that propagate along a horizontal rectangular channel, communicating to the liquid contained in this channel speeds that are roughly equal from the surface to the bottom). Lord Rayleigh [8], an English physicist, independently derived a similar equation in 1876, now known as the Rayleigh-Boussinesq equation. The renowned work done by Korteweg and de Vries [3] led to the derivation of the KdV equation, a simplified version of the Boussinesq equation, to model shallow water waves.

The Boussinesq equation has found applications in various fields, including shallow water waves and coastal engineering, nonlinear optics and fiber optics, and plasma physics and ion-acoustic waves.The generalized Boussinesq (GB) equation with a damping term is given by $u_{tt} = 2ku_{xxt} + qu_{xxxx} + c\,{u^n}_{xx}$, where k, q, c are constants and n is a nonzero real number[11]. This equation is widely used as a model to describe natural phenomena in many scientific fields, such as plasma waves, solid physics, and fluid mechanics.

A special case of the Boussinesq equation is the modified Boussinesq equation, which is obtained when k = 0, q = 1, c = -1, and n = 3. This equation is used to model the temporal evolution of nonlinear finite amplitude waves on a density front in a rotating fluid. Exact traveling wave solutions for the generalized Boussinesq equation have been studied using various methods, including the extended tanh method and the direct method. The Boussinesq equation appears in different forms, depending on the values of the constants k, q, c, and n. For example, if u_{xxt} is replaced by u_{xx} , the general form of the Boussinesq equation becomes$u_{tt} = u_{xx} + qu_{xxxx} + {u^2}_{xx}$ for n = 2, k = 1, and c = 1. This equation is known as the good Boussinesq or well-posed equation when q = -1, and the bad or ill-posed Boussinesq equation when q = 1.

Therefore, the equation under investigation in this dissertation is the ill-posed nonlinear PDE

$$u_{tt} = u_{xx} + {u^2}_{xx} + u_{xxxx} \qquad \ldots\ldots\ldots\ldots \quad (1.0)$$

This equation is characterized by its combination of second-order time derivatives and a mix of second and fourth-order spatial derivatives, along with a nonlinear term involving the square of the dependent variable. Such equations often arise in the study of wave propagation in nonlinear media, including the analysis of elastic waves, fluid dynamics, and other areas where higher-order dispersion effects and non-linearities are significant. Despite the challenges encountered in finding solutions to the (1.0), researchers have made significant progress in solving the ill-posed Boussinesq equation using various numerical methods. Recent advances in symmetry analysis and conservation laws through the multiplier method have provided new insights into solving the ill-posed Boussinesq equation[11,12]. This method has been successfully applied to other nonlinear PDEs, and researchers are hopeful that it will provide a breakthrough in solving the ill-posed Boussinesq equation.

1.1. Statement of problem

The ill-posed Boussinesq equation (1.0) poses significant challenges due to its non-integrable nature, which makes it difficult to find exact solutions. Furthermore, the equation's ill-posedness leads to numerical instability, making it challenging to obtain accurate numerical solutions. Due to the crucial role played by this equation, finding its exact solutions becomes significant as it throws more light into the physical features and intricate behavior of the system

1.2. Significance of the study

The significance of this study lies in its contribution to the theory and application of nonlinear PDEs. By constructing conservation laws and performing double reductions, this research advances the understanding of complex nonlinear systems and provides a methodology that can be applied to other PDEs with similar structures. The exact solutions obtained in this study offer valuable insights into the behavior of the equation and can serve as benchmarks for future analytical and numerical studies.

1.3. Scope of the study

This study is focused on the mathematical analysis of (1.0). The research is divided into four main components: the construction of conservation laws and Lie point symmetries, the reduction of the PDE using these laws, solving the reduced equations to get exact solutions and analysis of the solutions obtained

2. Methodology

lie point symmetry method were used for the derivation of the Lie symmetries while the conservation laws which serve as integral in-variants under the dynamics described by the PDE.associated are systematically constructed using multiplier method. We chose to use the multipliers method to construct the conservation laws of the equation (1.0) for the first time because of its numerous advantages over other methods. The method of multipliers helps to find conserved integrals and local continuity equations of PDES. The conserved vectors and lie symmetry vectors derived in turn serves as tools or approach to perform double reduction of the PDE under consideration

2.1. Lie point symmetry method

Lie point symmetries admitted by equation (1.0) are generated by a vector field of the form

$$V= \xi(x,t,u)\frac{\partial}{\partial x}+ \tau(x,t,u)\frac{\partial}{\partial t} +\phi(x,t,u)\frac{\partial}{\partial u} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.0)$$

and we need to solve for the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$, ϕ (x,t,u).

V must satisfy Lie's symmetry condition (1. 3), that is

$$V^{[4]}\,[u_{tt} - u_{xx} - u^2{}_{xx} - u_{xxxx=0}]|_{(1.0)} =0, \quad \ldots\ldots\ldots\ldots\ldots \quad (2.1)$$

where $V^{[4]}$ is the fourth prolongation of the operator V defined by

$$V^{[4]} =V+ \varsigma_x \frac{\partial}{\partial u_x} +\varsigma_t \frac{\partial}{\partial u_t}+ \varsigma_{xx} \frac{\delta}{\partial u_{xx}} + \varsigma_{tt} \frac{\partial}{\partial u_{tt}}+ \varsigma_{xx} \frac{\partial}{\partial u_x}+ \varsigma_{xxx} \frac{\partial}{\partial u_{xxx}}+\varsigma_{xxxx}\frac{\partial}{\partial u_{xxxx}}$$

and the coefficients ς_x, ς_t, ς_{xx}, ς_{tt}, ς_{xxx} and ς_{xxxx} are given by

$$\varsigma_x = D_x(\phi) - u_t D_x(\xi) - u_x D_x(\tau),$$

$$\varsigma_t = D_t(\phi) - u_t D_t(\xi) - u_x D_t(\tau),$$

$$\varsigma_{xx} = D_x(\zeta x) - u_{xt} D_x(\xi) - u_{xx} D_x(\tau),$$

$$\varsigma_{tt} = D_t(\zeta t) - u_{tt} D_x(\xi) - u_{tx} D_x((\tau),$$

$$\varsigma_{xxx} = D_x(\zeta_{xx}) - u_{xxt} D_x(\xi) - u_{xxx} D_x((\tau),$$

$$\varsigma_{xxxx} = D_x(\zeta_{xxx}) - U_{xxxt} D_x(\xi) - U_{xxxx} D_x((\tau).$$

Here D_x, D_t denote the total derivative operators defined by

$$D_x = \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{tx}\frac{\partial}{\partial u_t}+\ldots,\ D_t = \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{xt}\frac{\partial}{\partial u_x}+\ldots \quad \ldots\ldots\ldots \quad (2.2)$$

Expansion and separation of (2.1) with respect to the powers of different derivatives of u yields an over determined system in the unknown coefficients ξ, τ and ϕ. However the over determined system cannot be presented here due to its lengthy calculations. We present only the result and refer the reader to [8] for details. Solving the over determined system for arbitrary parameters we obtain thee coefficients

$$\xi(x,t,u) = \frac{1}{2}c_1 x + c_3, \tau(x,t,u) = c_1 t + c_2, \phi(x,t,u) = -c_1(u + \tfrac{1}{2}),$$

where c_1, c_2 and c_3 are constants. Without loss of generality, taking$c_3 = 1, c_1 = 0, c_2 = 0,$ we obtain the symmetry vector $V_1 = \frac{\partial}{\partial x}$.

Similarly, we obtain the rest as $V_2 = \frac{\partial}{\partial t}, V_3 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - (u + \frac{1}{2})\frac{\partial}{\partial u}$

These 3 symmetry vectors, v_1, v_2 and v_3 will be used to reduce equation (1.0) to simpler and solvable form.

2.2. Conservation laws via multiplier approach

A conserved vector corresponding to a conservation law of the equation (1.0) is a 2−tuple (T^t, T^x) such that

$$D_t T^t + D_x T^x = 0$$

along the solutions of the equation. We derive the conservation laws using the multiplier approach [9,10]

Consider the multiplier Λ of order up to two, viz. $\Lambda = (t, x, u, u_x, u_t, u_{xt}, u_{xx}, u_{tt}, u_{xxxx})$ for eqn.(1.0). The conserved vector (T^t, T^x) of eqn.(1.0) satisfies the divergence relation

$D_t T^t + D_x T^x = \Lambda(u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}) = 0$. Moreover, we have $\frac{\delta}{\delta u}(\Lambda)(u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}) = 0.$ (2.3)

After a lengthy calculation with the help of maple software we obtain the following conserved vectors with their corresponding multipliers:

$$\Lambda_1 = 1$$

$$T_1^t = -u_t, T_1^x = 2uu_x + u_x + u_{xxx}$$

$$\Lambda_2 = x$$

$$T_2^t = -xu_t, T_2^x = 2xuu_x - u^2 + xu_x + xu_{xxx} - u - u_{xx} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.4)$$

$$\Lambda_3 = t$$

$$T_3^t = -u_{tt} + u, T_3^x = 2tuu_x + tu_x + tu_{xxx}$$

$$\Lambda_4 = xt$$

$$T_4^t = -xtu_t + xu, T_4^x = 2xtuu_x - tu^2 + xtu_x + xtu_{xxx} - tu - tu_{xx}$$

2.3. Definition

Consider a scalar PDE F = 0 with n = 2,(x1,x2) = (t,x) which admits a symmetry X associated with a conserved vector ($T^{T,}$ T^X). In terms of the canonical variables r,s obtained by mapping X to Y = $\frac{\partial}{\partial s}$ the conservation laws can be expressed as[5]

$$D_r T^r + D_s T^s = 0,$$

with T^r and T^s given as

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)},$$

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}.$$

This now allows for a double reduction of the PDE.

2.4. Double reduction of equation (1.0)

We firstly show that the lie vector symmetries are associated with the conserved vectors. This happens if

$$V_i^{[3]}\begin{pmatrix}T_i^t\\T_i^x\end{pmatrix}+(D_t\xi+D_x\tau)\begin{pmatrix}T_i^t\\T_i^x\end{pmatrix}-\begin{pmatrix}T_i^t\\T_i^x\end{pmatrix}\begin{pmatrix}D_t\xi & D_x\xi\\D_t\tau & D_x\tau\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, i=1,2,3$$

T_i^x and T_i^t $i = 1, 2, 3$ and 4 are conserved vectors and $V_i^{[3]}$ is the third prolongation of the V_i , $i = 1, 2$ and 3

and $V_i^{[3]} = V_i + \varsigma_x \frac{\partial}{\partial u_x} + \varsigma_t \frac{\partial}{\partial u_t} + \varsigma_{xx} \frac{\partial}{\partial u_{xx}} + \varsigma_{tt} \frac{\partial}{\partial u_{tt}} + \varsigma_{xx} \frac{\partial}{\partial u_x} + \varsigma_{xxx} \frac{\partial}{\partial u_{xxx}}$

since V_1 and V_2 are trivial symmetries, they are associated with $T_1 = \begin{pmatrix}T_1^t\\T_1^x\end{pmatrix}$ with its multiplier Λ_1.

Next, we verify if V_3 is associated with $\begin{pmatrix}T_1^t\\T_1^x\end{pmatrix}$

$$V_3 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \left(u+\frac{1}{2}\right)\frac{\partial}{\partial u},$$

$$\xi(x,t,u) = \frac{1}{2}x, \tau(x,t,u) = t, \phi\,(x,t,u) = -(u+\tfrac{1}{2}).$$

$$D_t(\phi) = -u_t,, D_t(\tau) = 1,\ D_t(\xi) = 0, D_x(\phi) = -u_x, D_x(\xi) = \tfrac{1}{2}, D_x(\tau) = 0$$

$$\zeta_x = D_x(\phi) - u_tD_t(\xi) - u_xD_x(\tau) = -u_x - u_t.0 - u_x.0 = -u_x$$

$$\zeta_t = \zeta_t = D_t(\phi) - u_tD_t(\xi) - u_xD_t(\tau) = -u_t - u_t.0 - u_x.1 = -u_t - u_x$$

$$\zeta_{tt} = D_t(\zeta_t) - u_{tt}D_x(\xi) - u_{tx}D_x(\tau) = -u_{tt} - u_{tx} - \tfrac{1}{2}u_{tt} - u_{tx}.0 = -\tfrac{3}{2}u_{tt} - u_{tx}$$

$$\zeta_{xx} = D_x(\zeta_x) - u_{xt}D_x(\xi) - u_{xx}D_x(\tau) = -u_{xx} - \tfrac{1}{2}u_{xt} - u_{xx}.0 = -u_{xx} - \tfrac{1}{2}u_{xt}$$

$$\zeta_{xxx} = D_x(\zeta_{xx}) - u_{xxt}D_x(\xi) - u_{xxx}D_x(\tau) = -u_{xxx} - \tfrac{1}{2}u_{xxt} - \tfrac{1}{2}u_{xxt} - \tfrac{1}{2}u_{xxx}.0 = -u_{xxx} - u_{xxt}$$

Substituting (1.) we obtain the third prolongation of V_3 as

$$V_3^{(3)} = \tfrac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - (u+\tfrac{1}{2})\frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_x} + (-u_t - u_x)\frac{\partial}{\partial u_t} + (-u_{xx} - \tfrac{1}{2}u_{xt})\frac{\delta}{\partial u_{xx}} + (-\tfrac{3}{2}u_{tt} - u_{xt})\frac{\partial}{\partial u_{tt}} + (-u_{xx} - \tfrac{1}{2}u_{xt})\frac{\partial}{\partial u_x} + (-u_{xxx} - u_{xxt})\frac{\partial}{\partial u_{xxx}}$$

Also substituting into

$$\left(\frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - (u+\frac{1}{2})\frac{\partial}{\partial u} - ux\frac{\partial}{\partial u_x} + (-ut - ux)\frac{\partial}{\partial u_t} + (-uxx - \frac{1}{2}uxt)\frac{\delta}{\partial u_{xx}} + (-\frac{3}{2}u_{tt} - u_{xt})\frac{\partial}{\partial u_{tt}} + (-uxx - \frac{1}{2}uxt)\frac{\partial}{\partial u_x} + (-uxxx - uxxt)\frac{\partial}{\partial u_{xxx}}\right)\begin{pmatrix}-u_t\\2uu_x+u_x+u_{xxx}\end{pmatrix} + \left(\frac{1}{2} \quad + \quad 1\right)\begin{pmatrix}-u_t\\2uu_x+u_x+u_{xxx}\end{pmatrix} - \begin{pmatrix}1 & 0\\0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-u_t\\2uu_x+u_x+u_{xxx}\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$$

Therefore V_3 is not associated with $\begin{pmatrix}T_1^t\\T_1^x\end{pmatrix}$ with multiplier Λ_1. The table below shows the relationship between the 3 vector symmetries and the conserved vectors with their multipliers.

Table 1 Relationship between the point symmetry vectors and conserved vectors

Multipliers	Conserved Vectors	Symmetry Vectors	Association
$\Lambda_1 = 1$	$T_1^t,\ T_1^x$	V_1 , V_2 , V_3	x
$\Lambda_2 = x$	$T_2^t,\ T_2^x$	V_1 , V_2 , V_3	x
$\Lambda_3 = t$	$T_3^t,\ T_3^x$	V_1 , V_2 , V_3	x
$\Lambda_4 = xt$	$T_4^t,\ T_4^x$	V_1 , V_2, V_3	x

We now use the lie point symmetries which are associated with the conserved vector $\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix}$ to transform the variables of the Boussinesq equation (1.0) into new similarity variables. we consider the linear combination V_2 + cV_1 where c is a non - zero constant. solving the characteristic equation.

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}$$

we obtain the canonical coordinates

$$s = t, r = x - ct, w\ (r) = u$$

In the new canonical coordinates, the conservation law $D_t\, T_1^t + D_x\, T_1^s = 0$ is rewritten as

$$D_r\, T_1^r + D_s\, T_1^s = 0$$

We can find T_1^r and T_1^s by

$$T_1^r = \frac{T_1^t D_t\,(r) + T_1^x D_x\,(r)}{D_t\,(r)\,D_x\,(s) - D_x\,(r)\,D_t\,(s)} \qquad \ldots\ldots\ldots\ldots\ldots \quad (2.5)$$

$$T_1^s = \frac{T_1^t D_t\,(s) + T_1^x D_x\,(s)}{D_t\,(r)\,D_x\,(s) - D_x\,(r)\,D_t\,(s)}$$

Substituting the canonical variables, their derivatives and the conserved vectors into (2.5) we obtain

$$T_1^r = -cu_t - 2uu_x - u_x - u_{xxx} \qquad \ldots\ldots\ldots\ldots \ (2.6)$$

$$T_1^s = -u_t$$

Now,

$$u_t = \frac{du}{dt} = \frac{dw}{dt} = \frac{\partial w}{\partial r}.\frac{\partial r}{\partial t} = -\,cw_r$$

$$u_x = \frac{du}{dx} = \frac{dw}{dx} = \frac{\partial w}{\partial r}.\frac{\partial r}{\partial x} = 1.\,w_r = w_r$$

Similarly,

$$u_{xx} = \frac{du_x}{dx}\ \frac{dw_r}{dx} = \frac{\partial w_r}{\partial r}.\frac{\partial r}{\partial x} = w_{rr}$$

$$u_{xxx} = w_{rrr}$$

using (2.6) we obtain

$$T_1^r = (1 - c^2)w_r + 2ww_r + w_{rrr} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2.7)$$

$$T_1^s = cw_r \qquad \ldots\ldots\ldots\ldots\ldots\ldots \ (2.8)$$

Where w_r is total derivative with respect to r. Since (2.8) does not depend on s, we deduce from that

$$D_r\, T_1^r = 0$$

This means that T_1^r is a constant. That is m

$$(1-c^2)w_r + 2ww_r + w_{rrr} = k_1 \qquad \text{.................} \qquad (2.9)$$

where k_1 is a constant. Equation (2.9) is a third order ODE which is a double reduction of the fourth order ill-posed boussinesq equation (1.0).

3. Results and Discussions

By integrating (2.9) once with respect to r, while setting the constant of integration to zero gives rise to

$$(1-c^2)w + w^2 + w_{rr} = 0$$

Further integration results to

$$(1\text{-}c^2)\frac{w^2}{2} + \frac{w^3}{3} + \frac{{w_r}^2}{2} = k_3\text{w}$$

where k_3 is a constant. This implies

$${w_r}^2 = (c^2-1)w^2 - \frac{2\,w^3}{3} + 2\text{w}k_3$$

$$w_r = \sqrt{(c^2-1)w^2 - \frac{2\,w^3}{3} + 2\text{w}k_3}$$

$$\frac{dw}{dr} = \sqrt{(c^2-1)w^2 - \frac{2\,w^3}{3} + 2\text{w}k_3}$$

$$\frac{dr}{dw} = \frac{1}{\sqrt{(c^2-1)w^2 - \frac{2\,w^3}{3} + 2\text{w}k_3}}$$

$$\frac{dw}{\sqrt{(c^2-1)w^2 - \frac{2\,w^3}{3} + 2\text{w}k_3}} = \text{dr}$$

$$\int \frac{dw}{\sqrt{(c^2-1)w^2 - \frac{2\,w^3}{3} + 2\text{w}k_3}} = \text{r} + k_4$$

where k_4 is constant. In terms of the original variables, we obtain

$$\int \frac{du}{\sqrt{(c^2-1)u^2 - \frac{2}{3}u^3 + 2\text{u}k_3}} = x - ct + k_4 \qquad \text{..............} \; (3.0)$$

Equation (1.12) is the integral solution of the ill-posed Boussinesq equation (1.0).

4. Exact solutions by improved generalised Riccati equation mapping method

Here, we solve the reduced equation (2.9) using improved generalized Riccati equation mapping method [4]. Our main aim is to obtain exact or at least approximate solutions if possible for the reduced equation (2.9). We express the solution, w(r) of equation (2.9) in the finite series

$$\text{w(r)} = \sum_{i=-m}^{m} a_i \psi^i, \qquad \text{..................} \quad (3.1)$$

where a_i are constants to be determined and ψ satisfies the Riccati equation

$$\psi^I = \mu + \beta\psi + (\nu - 1)\psi^2 \qquad \ldots\ldots\ldots\ldots \quad (3.2)$$

We determine the positive integer m in equation (3.1) by balancing the highest order derivative, w_{rr} and the nonlinear term, w^2 by solving

$$m + 2 = 2m \Rightarrow m = 2,$$

so that the solution of equation (4.36) can be written as

$$\mathrm{w(r)} = a_{-2}\psi^{-2} + a_{-1}\psi^{-1} + a_0 + a_1\psi + a_2\psi^2 \qquad \ldots\ldots\ldots \quad (3.3)$$

After substituting and collecting all the terms of the same power, $\psi^i, i = -2, -1, 0, 1, 2$ and equating them to zero, we obtain a system of an algebraic equations (Due to the size of the equations we decided not to display the equation for simplicity). Solving the system of the algebraic equations for $a_{-2}, a_{-1}, a_0, a_1, a_2, c$, using symbolic computation software, Mathematica 9, we obtain

$$a_1 = a_2 = 0, a_0 = -6\mu(\nu - 1), a_{-1} = -6\beta\mu, a_{-2} = -6\mu^2 \qquad \ldots\ldots\ldots\ldots \quad (3.4)$$

Substituting equation (3.4) into the solution formula (3.3), we obtain

$$\mathrm{w(r)} = -6\mu^2\psi^{-2} - 6\beta\mu\psi^{-1} - 6\mu(\nu - 1) \qquad \ldots\ldots\ldots\ldots\ldots (3.5)$$

Substituting the known solutions, $\psi(r)$ of the Riccati equation (3.2) into equation (3.5) and simplifying the resulting equation in terms of the original variable, $u(x,t)$, we obtained the following new types of solutions:

TYPE 1: $\Omega = \beta^2 - 4\mu(\nu - 1) > 0, \beta(\nu - 1) \neq 0, (or\ \mu(\nu - 1) \neq 0)$, Soliton like solutions

$$u_1(x,t) = -6\mu(\nu - 1) + 12\beta\mu(\nu - 1)\left[\beta + \sqrt{\Omega}\,Tanh\left(\frac{\sqrt{\Omega}}{2}(x - \lambda t)\right)\right]^{-1}$$

$$-24(\mu(\nu - 1))^2\left[\beta + \sqrt{\Omega}\,Tanh\left(\frac{\sqrt{\Omega}}{2}(x - \lambda t)\right)\right]^{-2}$$

$$u_2(x,t) = -6\mu(\nu - 1) + 12\beta\mu(\nu - 1)\left[\beta + \sqrt{\Omega}\,Coth\left(\frac{\sqrt{\Omega}}{2}(x - \lambda t)\right)\right]^{-1}$$

$$-24(\mu(\nu - 1))^2\left[\beta + \sqrt{\Omega}\,Coth\left(\frac{\sqrt{\Omega}}{2}(x - \lambda t)\right)\right]^{-2}$$

$$u_3(x,t) = -6\mu(\nu - 1) + 12\beta\mu(\nu - 1)\left[\beta + \sqrt{\Omega}\left(Tanh\left(\sqrt{\Omega}(x - \lambda t)\right) \pm iSech\left(\sqrt{\Omega}(x - \lambda t)\right)\right)\right]^{-1} - 24(\mu(\nu - 1))^2\left[\beta + \sqrt{\Omega}\left(Tanh\left(\sqrt{\Omega}(x - \lambda t)\right) \pm iSech\left(\sqrt{\Omega}(x - \lambda t)\right)\right)\right]^{-2}$$

$$u_4(x,t) = -6\mu(\nu - 1) + 12\beta\mu(\nu - 1)\left[\beta + \sqrt{\Omega}\left(Coth\left(\sqrt{\Omega}(x - \lambda t)\right) \pm Csch\left(\sqrt{\Omega}(x - \lambda t)\right)\right)\right]^{-1} - 24(\mu(\nu - 1))^2\left[\beta + \sqrt{\Omega}\left(Coth\left(\sqrt{\Omega}(x - \lambda t)\right) \pm Csch\left(\sqrt{\Omega}(x - \lambda t)\right)\right)\right]^{-2}$$

$$u_5(x,t) = -6\mu(\nu - 1) + 24\beta\mu(\nu - 1)\left[2\beta + \sqrt{\Omega}\left(Tanh\left(\frac{\sqrt{\Omega}}{4}(x - \lambda t)\right) \pm Coth\left(\frac{\sqrt{\Omega}}{4}(x - \lambda t)\right)\right)\right]^{-1} - 96(\mu(\nu - 1))^2\left[2\beta + \sqrt{\Omega}\left(Tanh\left(\frac{\sqrt{\Omega}}{4}(x - \lambda t)\right) \pm Coth\left(\frac{\sqrt{\Omega}}{4}(x - \lambda t)\right)\right)\right]^{-2}$$

$$u_6(x,t) = -6\mu(\nu-1) - 12\beta\mu(\nu-1)\left[-\beta + \frac{\sqrt{(A^2+B^2)\Omega} - A\sqrt{\Omega}\,Cosh\left(\sqrt{\Omega}\,(x-\lambda t)\right)}{A\,Sinh\left(\sqrt{\Omega}\,(x-\lambda t)\right)+B}\right]^{-1}$$

$$-24(\mu(\nu-1))^2\left[-\beta + \frac{\sqrt{(A^2+B^2)\Omega} - A\sqrt{\Omega}\,Cosh\left(\sqrt{\Omega}\,(x-\lambda t)\right)}{A\,Sinh\left(\sqrt{\Omega}\,(x-\lambda t)\right)+B}\right]^{-2}$$

$$u_7(x,t) = -6\mu(\nu-1) - 12\beta\mu(\nu-1)\left[-\beta - \frac{\sqrt{(A^2+B^2)\Omega} + A\sqrt{\Omega}\,Cosh\left(\sqrt{\Omega}\,(x-\lambda t)\right)}{A\,Sinh\left(\sqrt{\Omega}\,(x-\lambda t)\right)+B}\right]^{-1}$$

$$-24(\mu(\nu-1))^2\left[-\beta - \frac{\sqrt{(A^2+B^2)\Omega} + A\sqrt{\Omega}\,Cosh\left(\sqrt{\Omega}\,(x-\lambda t)\right)}{A\,Sinh\left(\sqrt{\Omega}\,(x-\lambda t)\right)+B}\right]^{-2}$$

where A and B are two non-zero constants and satisfies $B^2 - A^2 > 0$.

$$u_8(x,t) = -6\mu(\nu-1) - 3\beta\left[\frac{Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\sqrt{\Omega}\;Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) - \beta Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-1}$$

$$-\frac{3}{2}\left[\frac{Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\sqrt{\Omega}\;Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) - \beta Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-2}$$

$$u_9(x,t) = -6\mu(\nu-1) + 3\beta\left[\frac{Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\beta\;Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) - \sqrt{\Omega}\;Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-1}$$

$$-\frac{3}{2}\left[\frac{Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\beta\;Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) - \sqrt{\Omega}\;Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-2}$$

$$u_{10}(x,t) = -6\mu(\nu-1) - 3\beta\left[\frac{Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\sqrt{\Omega}\;Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) - \beta Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) \pm i\sqrt{\Omega}}\right]^{-1}$$

$$-\frac{3}{2}\left[\frac{Cosh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\sqrt{\Omega}\;Sinh\left(\sqrt{\Omega}(x-\lambda t)\right) - \beta Cosh\left(\sqrt{\Omega}(x-\lambda t)\right) \pm i\sqrt{\Omega}}\right]^{-2}$$

$$u_{11}(x,t) = -6\mu(\nu-1) - 3\beta\left[\frac{Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{-\beta\, Sinh\left(\sqrt{\Omega}(x-\lambda t)\right) + \sqrt{\Omega}\; Cosh\left(\sqrt{\Omega}(x-\lambda t)\right) \pm \sqrt{\Omega}}\right]^{-1}$$

$$-\frac{3}{2}\left[\frac{Sinh\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{-\beta\, Sinh\left(\sqrt{\Omega}(x-\lambda t)\right) + \sqrt{\Omega}\; Cosh\left(\sqrt{\Omega}(x-\lambda t)\right)} \pm \sqrt{\Omega}\right]^{-2}$$

$$u_{12}(x,t) = -6\mu(\nu-1) - \frac{3}{2}\beta\left[\frac{Sinh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) Cosh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right)}{-2\beta Sinh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) Cosh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) + 2\sqrt{\Omega}\, Cosh^2\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) - \sqrt{\Omega}}\right]^{-1}$$

$$-\frac{3}{8}\left[\frac{Sinh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) Cosh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right)}{-2\beta Sinh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) Cosh\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) + 2\sqrt{\Omega}\, Cosh^2\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) - \sqrt{\Omega}}\right]^{-2}$$

TYPE 2:$\Omega = \beta^2 - 4\mu(\nu-1) < 0, \beta(\nu-1) \neq 0, (or\ \mu(\nu-1) \neq 0$,Periodic form solutions

$$u_{13}(x,t) = -6\mu(\nu-1) - 12\beta\mu(\nu-1)\left[-\beta + \sqrt{\Omega}\, Tan\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)\right]^{-1}$$

$$-24(\mu(\nu-1))^2\left[-\beta + \sqrt{\Omega}\, Tan\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)\right]^{-2}$$

$$u_{14}(x,t) = -6\mu(\nu-1) + 12\beta\mu(\nu-1)\left[\beta + \sqrt{\Omega}\, Cot\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)\right]^{-1}$$

$$-24(\mu(\nu-1))^2\left[\beta + \sqrt{\Omega}\, Cot\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)\right]^{-2}$$

$$u_{15}(x,t) = -6\mu(\nu-1) - 12\beta\mu(\nu-1)\left[-\beta + \sqrt{\Omega}\left(Tan\left(\sqrt{\Omega}(x-\lambda t)\right) \pm iSec\left(\sqrt{\Omega}(x-\lambda t)\right)\right)\right]^{-1} -24(\mu(\nu-1))^2\left[-\beta + \sqrt{\Omega}\left(Tan\left(\sqrt{\Omega}(x-\lambda t)\right) \pm iSec\left(\sqrt{\Omega}(x-\lambda t)\right)\right)\right]^{-2}$$

$$u_{16}(x,t) = -6\mu(\nu-1) + 12\beta\mu(\nu-1)\left[\beta + \sqrt{\Omega}\left(Cot\left(\sqrt{\Omega}(x-\lambda t)\right) \pm Csc\left(\sqrt{\Omega}(x-\lambda t)\right)\right)\right]^{-1} -24(\mu(\nu-1))^2\left[\beta + \sqrt{\Omega}\left(Cot\left(\sqrt{\Omega}(x-\lambda t)\right) \pm Csc\left(\sqrt{\Omega}(x-\lambda t)\right)\right)\right]^{-2}$$

$$u_{17}(x,t) = -6\mu(\nu-1) - 24\beta\mu(\nu-1)\left[-2\beta + \sqrt{\Omega}\left(Tan\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) - Cot\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right)\right)\right]^{-1} -96(\mu(\nu-1))^2\left[-2\beta + \sqrt{\Omega}\left(Tan\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right) - Cot\left(\frac{\sqrt{\Omega}}{4}(x-\lambda t)\right)\right)\right]^{-2}$$

$$u_{18}(x,t) = -6\mu(\nu-1) - 12\beta\mu(\nu-1)\left[-\beta + \frac{\pm\sqrt{(A^2+B^2)\Omega} - A\sqrt{\Omega}\,Cos\left(\sqrt{\Omega}(x-\lambda t)\right)}{A\,Sin\left(\sqrt{\Omega}(x-\lambda t)\right) + B}\right]^{-1}$$

$$-24(\mu(\nu-1))^2\left[-\beta + \frac{\pm\sqrt{(A^2+B^2)\Omega} - A\sqrt{\Omega}\,Cos\left(\sqrt{\Omega}(x-\lambda t)\right)}{A\,Sin\left(\sqrt{\Omega}(x-\lambda t)\right) + B}\right]^{-2}$$

$$u_{19}(x,t) = -6\mu(\nu-1) - 12\beta\mu(\nu-1)\left[-\beta - \frac{\pm\sqrt{(A^2+B^2)\Omega} + A\sqrt{\Omega}\,Cos\left(\sqrt{\Omega}(x-\lambda t)\right)}{A\,Sin\left(\sqrt{\Omega}(x-\lambda t)\right) + B}\right]^{-1}$$

$$-24(\mu(\nu-1))^2\left[-\beta - \frac{\pm\sqrt{(A^2+B^2)\Omega} + A\sqrt{\Omega}\,Cos\left(\sqrt{\Omega}(x-\lambda t)\right)}{A\,Sin\left(\sqrt{\Omega}(x-\lambda t)\right) + B}\right]^{-2}$$

where A and B are two non-zero constants and satisfies $A^2 - B^2 > 0$

$$u_{20}(x,t) = -6\mu(\nu-1) - 3\beta\left[\frac{Cos\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\sqrt{\Omega}\;Sin\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) + \beta Cos\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-1}$$

$$-\frac{3}{2}\left[\frac{Cos\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{\sqrt{\Omega}\;Sin\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) + \beta Cos\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-2}$$

$$u_{21}(x,t) = -6\mu(\nu-1) + 3\beta\left[\frac{Sin\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{-\beta\;Sin\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) + \sqrt{\Omega}\;Cos\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-1}$$

$$-\frac{3}{2}\left[\frac{Sin\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}{-\beta\;Sin\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right) + \sqrt{\Omega}\;Cos\left(\frac{\sqrt{\Omega}}{2}(x-\lambda t)\right)}\right]^{-2}$$

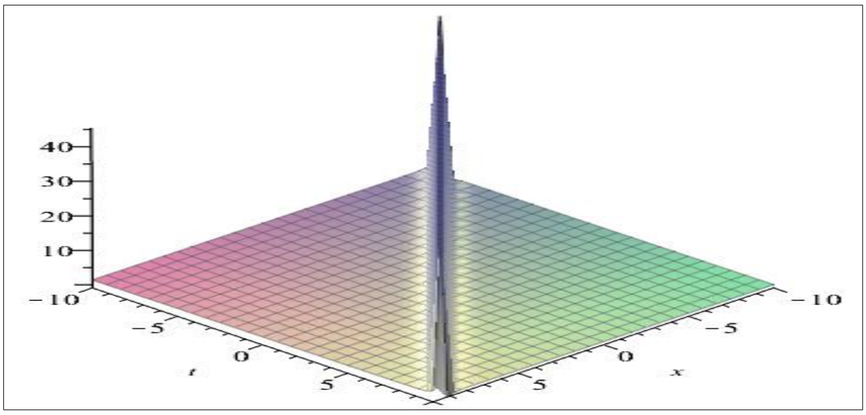

Figure 1 Graph of the cuspon solution $\boldsymbol{u_1(x,t)}$ for $\boldsymbol{\beta}$= 1, $\boldsymbol{\mu}$= -1, v = 2 with -10≤ $\boldsymbol{x,t}$≤ 10

Solutions $u_{23}(x,t)$and $u_{24}(x,t)$are bell shaped sech[2] solitary traveling wave solutions. Solutions $u_2(x,t)$ and $u_4(x,t)$are singular soliton solutions. Figure 2 shows the shape of exact soliton traveling wave solution $u_2(x,t)$ of equation (4.36). The shape of figures of the solution $u_4(x,t)$ is similar to $u_2(x,t)$. The solution$u_5(x,t)$is a singular kink solution. Figure 3 shows the shape of exact singular kink-type solution (shown here is only the shape of the solution $u_5(x,t)$with β= 1, μ= -1, v = 2 with -10≤ x,t≤ 10

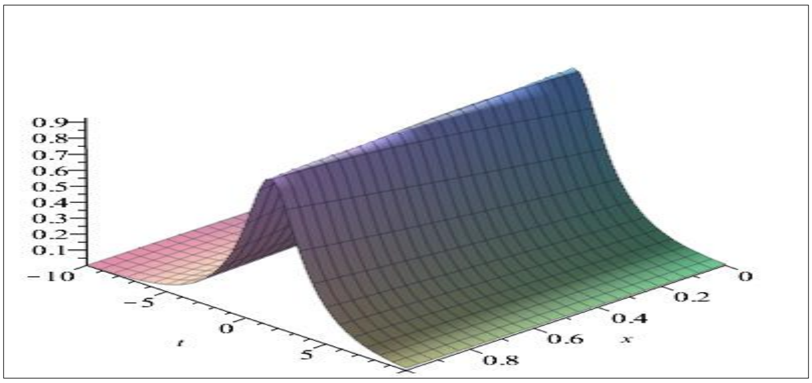

Figure 2 Graph of the soliton traveling wave solution $\boldsymbol{u_6(x,t)}$for $\boldsymbol{\beta}$= 1, $\boldsymbol{\mu}$= -1, v = 2 with -10≤ $\boldsymbol{x,t}$≤ 10

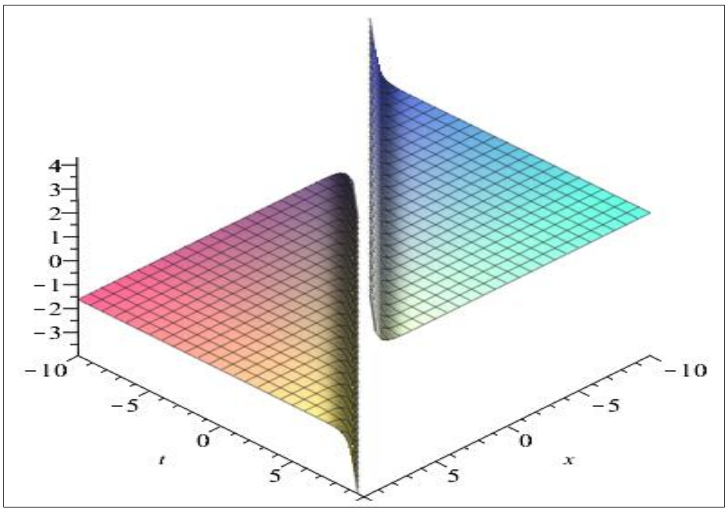

Figure 3 Graph of singular kink traveling wave solution $\boldsymbol{u_5(x,t)}$for $\boldsymbol{\beta}$= 1, $\boldsymbol{\mu}$= -1, v = 2 with -10≤ $\boldsymbol{x},\boldsymbol{t}$≤ 10

Solutions $u_6(x,t)$ to $u_{11}(x,t)$, $u_{23}(x,t)$and $u_{24}(x,t)$describe the soliton, which is a special type of solitary wave, The soliton solution is specially localized solution; hence u',u" tends to zero as ξ tends to ±∞ and ξ=$x-ct.$ The soliton has a remarkable property in that it keeps its identity upon interaction with other solutions. Solutions $u_{12}(x,t)$, $u_{14}(x,t)$, $u_{15}(x,t)$, $u_{17}(x,t)$, $u_{19}(x,t)$ and $u_{21}(x,t)$represent the exact traveling wave solution. The periodic solutions are exact traveling wave solution that are periodic such as $\cos(x-t)$. Figure 4 below shows the periodic solution u_{12}(x,t). The graph of the periodic solution $u_{12}(x,t)$ for v = 1β= 1, μ= -1, v = 2 with -1≤ x,t≤ 1 is omitted for convenience. Solutions $u_{13}(x,t)$, $u_{16}(x,t)$, $u_{18}(x,t)$, $u_{20}(x,t)$, $u_{22}(x,t)$ are exact singular period solutions. Figure 5 shows the shape of $u_{13}(x,t)$ with v = 1,β= 1, μ= 1, v = 2 and -1≤ x,t≤ 1

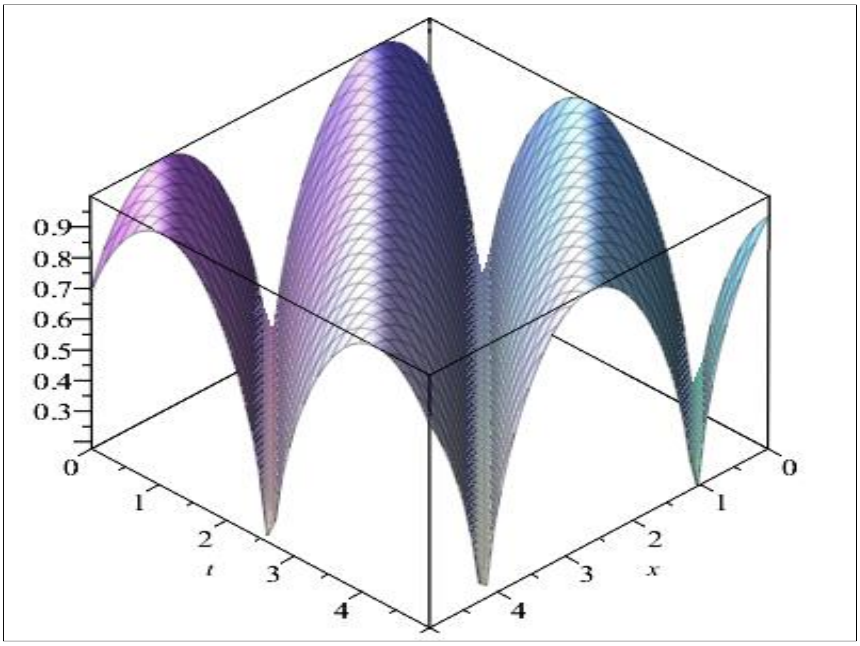

Figure 4 Graph of the periodic traveling wave solution $\boldsymbol{u_{12}(x,t)}$ for $\boldsymbol{\beta}$= 1, $\boldsymbol{\mu}$= -1, v = 2 with -10≤ $\boldsymbol{x},\boldsymbol{t}$≤ 10

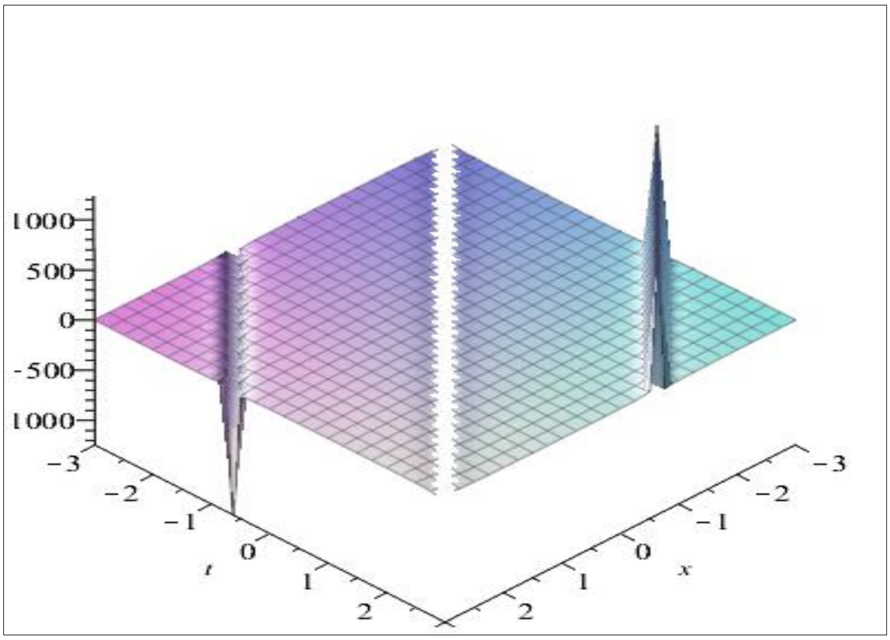

Figure 5 Graph of the singluar periodic traveling wave solution $\boldsymbol{u_{13}}(\boldsymbol{x},\boldsymbol{t})$

5. Conclusion

The symmetry generators and conservation laws derived using the multiplier method were used to reduce the original fourth-order Boussinesq equation to a second-order ordinary differential equation (ODE). This was achieved by identifying the symmetry variables and using them to reduce the order of the equation. The reduced second-order ODE was solved exactly, yielding new solutions to the ill-posed Fourth-order Boussinesq PDE. The exact solutions obtained provide insight into the physical behavior of the system modeled by the equation and can be used to verify the accuracy of numerical simulations, study the stability, chaotic and asymptotic behavior of the system. The technique used helps to advances the understanding of complex nonlinear systems and provides a methodology that can be applied to other PDEs with similar structures. The exact solutions also offer valuable insights into the behavior of the equation and can serve as benchmarks for future analytical and numerical studies. The exact solutions obtained were graphically displayed and analyzed, providing a visual representation of the behavior of the solutions. It revealed the structure of the solutions, including the formation of solitary waves and shock waves, and also the effect of the parameters on the behavior of the solutions.

Recommendations

Based on the findings of this research, we recommend that symmetry analysis and conservation law derivation techniques developed in this research can be applied to other nonlinear partial differential equations, and simulations can be performed to verify the accuracy of the exact solutions obtained in this research and to explore the behavior of the equation in different regimes. Furthermore future research should aim to generalize the methodology employed in this study to encompass all nonlinear PDEs that lack lie point symmetries and conserved vectors, thereby broadening its applicability and scope.

Compliance with ethical standards

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Disclosure of conflict of interest

No conflict of interest to be disclosed.

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