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(Research Article)

Bulk viscous Kantowaski-Sachs Space-time with $H(z)$ parameterization

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Abstract

In this paper, we investigate the Kantowski-Sachs cosmological model with a perfect fluid within the framework of Teleparallel gravity, specifically $f(T)$ gravity, where T represents the torsion scalar. The behavior of the accelerating universe is explored by adopting a specific form of $f(T) = T^{\beta}$ and establishing a relationship between the metric potentials, $A = B^{n}$. By employing the Hubble parameterization method, we derive an exact solution to the field equations that supports an accelerating universe. The physical behavior of the model is analyzed through key physical quantities, and the functional dependence of the torsion scalar on the evolution of the universe is also evaluated.

Keywords: Kantowaski Sachs Universe; F(T) Theory of Gravity; Exact Solution; Cosmology

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1. Introduction

Theoretical work by Misner (1968) and recent observational data suggest that the universe exhibits isotropy and homogeneity on large scales in its current state. However, this may not have been the case in the past. To accurately describe the universe's evolution, models that incorporate anisotropic backgrounds, which transition to isotropy at later times, are particularly relevant. The Kantowski-Sachs spacetime, characterized by spatial homogeneity and anisotropy, offers a suitable framework for exploring the universe's evolution. Due to its potential significance in early cosmology, researchers have sought to investigate the properties of this spacetime in greater detail.

Recent cosmological observations, including those from Type-Ia Supernovae, cosmic microwave background radiation, large-scale structure, and the Wilkinson Microwave Anisotropy Probe [1-5], collectively suggest that our universe is spatially flat and comprised of approximately 70% dark energy (DE). This mysterious component is characterized by negative pressure, driving the accelerating expansion of the universe. While the cosmological constant Λ is a simple candidate for DE, it is plagued by theoretical issues such as fine-tuning and coincidence problems [6]. Alternative approaches to addressing the acceleration problem involve modifying the gravity law, as seen in $f(R)$ modified gravity [7-10], Gauss-Bonnet gravity [11-17], and Teleparallel Gravity [18-19]. The latter, also known as $f(T)$ theory, utilizes the Weitzenbock connection instead of the Levi-Civita connection, resulting in a theory with torsion but no curvature. This approach has garnered significant attention, with various studies exploring its cosmological implications [20-31]. Notably, researchers have proposed new $f(T)$ models, analyzed their observational viability, and investigated their dynamical properties, including phase space analysis and stability of critical points [32-36]. These efforts continue to advance our understanding of the universe's accelerating expansion and the role of dark energy.

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2. The action by generalizing the Teleparallel Theory i.e. F(T) theory as [18]

$$S = \int [T + f(T) + L_{matter}]\ e\, d^4x. \qquad \text{..................(1)}$$

Here,f(T) denotes an algebraic function of the torsion scalar T. The line element of the Riemannian manifold is given by

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu. \qquad \text{.................. (2)}$$

This line element can be converted to the Minkowskian description of the transformation called tetrad, as follows

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij}\theta^i\theta^j, \qquad \text{.................. (3)}$$

$$dx^\mu = e_i^\mu \theta^i,\ \ \theta^i = e_\mu^i dx^\mu, \qquad \text{.................. 4)}$$

where$\eta_{ij} = diag[1,-1,-1,-1]$ and $e_i^\mu e_\mu^i = \delta_\nu^\mu$ or $e_i^\mu e_\mu^j = \delta_i^j$.

The root of metric determinant is given by$\sqrt{-g} = det[\ e_\mu^i] = e$. For a manifold in which the Riemann tensor part without the torsion terms is null (contribution of the Levi-Civita connection) and only the non-zero torsion terms exist, the Weitzenbocks connection components are defined as

$$\Gamma_{\mu\nu}^\alpha = e_i^\alpha \partial_\nu e_\mu^i = -e_\mu^i \partial_\nu e_i^\alpha. \qquad \text{.................. (5)}$$

which has a zero curvature but nonzero torsion. Through the connection, we can define the components of the torsion tensors as

$$T_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - \Gamma_{\nu\mu}^\alpha = e_i^\alpha\left(\partial_\mu e_\nu^i - \partial_\mu e_\mu^i\right), \qquad \text{.................. (6)}$$

The difference between the Levi-Civita and Weitzenbock connections is a space-time tensor, and is known as the contorsion tensor:

$$K_\alpha^{\mu\nu} = \left(-\tfrac{1}{2}\right)\left(T^{\mu\nu}{}_\alpha + T^{\nu\mu}{}_\alpha - T_\alpha^{\mu\nu}\right). \qquad \text{.................. (7)}$$

For facilitating the description of the Lagrangian and the equations of motion, we can define another tensor $S_\alpha^{\mu\nu}$ from the components of the torsion and contorsion tensors, as

$$S_\alpha^{\mu\nu} = \left(\tfrac{1}{2}\right)\left(K^{\mu\nu}{}_\alpha + \delta_\alpha^\mu T^{\beta\nu}{}_\beta - \delta_\alpha^\nu T_\beta^{\beta\mu}\right). \qquad \text{.................. (8)}$$

The torsion scalar $T$is

$$T = T_{\mu\nu}^\alpha\, S_\alpha^{\mu\nu}. \qquad \text{..................9)}$$

Making the functional variation of the action (1) with respect to the tetrads, we get the following equations of motion

$$S_\mu^{\nu\rho}\partial_\rho T f_{TT} + \left[e^{-1}e_\mu^i \partial_\rho\left(e e_i^\alpha S_\alpha^{\nu\rho}\right) + T^\alpha{}_{\lambda\mu} S_\alpha^{\nu\lambda}\right](f_T) + \tfrac{1}{4}\delta_\mu^\nu (f) = 4\pi T_\mu^\nu, \qquad \text{.................. (10)}$$

The field equation (10) is written in terms of the tetrad and partial derivatives and appears very different from Einstein's equations.

whereT_μ^ν is the energy momentum tensor, $f_T = df(T)/dT$ and by setting $f(T) = a_0 =$constant this is dynamically equivalent to the GR.

Motivated by the preceding discussions, this paper investigates viscous fluid solutions in Kantowski-Sachs space-time within the framework of exponential $f(T)$ gravity. The paper's structure is as follows: Section 2 derives the field equations for $f(T)$ gravity. Section 3 presents an exact singular solution to the field equations and examines the physical implications of this solution. Finally, Section 4 summarizes the key findings and provides concluding remarks.

3. Field equations and some physical quantities

The line element of Kantowaski-Sachs space-time is given by

$$ds^2 = dt^2 - A^2dr^2 - B^2(d\theta^2 + sin^2\theta\, d\varphi^2), \qquad \text{..................} (11)$$

where the metric potentials A and B be the functions of time t only.

The corresponding Torsion scalar is given by

$$T = -2\left(2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2}\right). \qquad \text{..................} (12)$$

The energy momentum tensor T_j^i for the perfect fluid distribution is taken as

$$T_\mu^\nu = (\bar{p} + \rho)u^\nu u_\mu - \bar{p}g_\mu^\nu, \qquad \text{..................} (13)$$

atisfying the equation of state

$$\bar{p} = p - 3H\xi, \qquad \text{..................} (14)$$

together with commoving co-ordinates

$$u^\nu = (0,0,0,1) \text{ and } u^\nu u_\nu = 1, \qquad \text{..................} \qquad (15)$$

where u^ν is the 4-velocity vector of the cosmic fluid, $\bar{p}$, p and ρ be the effective pressure, anisotropic pressure and energy density of the fluid respectively.

From the equation of motion (10), Kantowaski-Sachs space-time (11) for the fluid of stress energy tensor (13) can be written as

$$f + 4(f_T)\left\{\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\} + 4\frac{\dot{B}}{B}\dot{T}f_{TT} = 16\pi(-\bar{p}), \qquad \text{..................} (16)$$

$$f + 2(f_T)\left\{\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + 3\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\} + 2\left\{\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right\}\dot{T}f_{TT} = 16\pi(-\bar{p}), \text{..................} (17)$$

$$f + 4(f_T)\left\{\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\} = 16\pi(\rho). \qquad \text{..................} (18)$$

where the dot $(\cdot)$ denotes the derivative with respect to time t. For the sake of simplicity, we consider the model of the $f(T)$ gravity as $f(T) = T^\beta$, where β be any model parameter.

Finally, here we have three differential equations with six unknowns namely $A, B, f, p, \bar{p}, \rho$. To find the solution of the field equation we need the extra conditions. Hence first we consider

The relation between the metric potentials as

$$A = B^n, \qquad \text{..................} (19 \text{ and}$$

The Hubble parameter H of the form

$$H = \frac{H_0}{\sqrt{2}}(1 + (1+z)^{2+2\alpha})^{\frac{1}{2}}. \qquad \text{..................} (20)$$

Now, we define some kinematical quantities that are related to the space-time such as

For this model, the corresponding metric coefficients A and B with the help of equations (19), and (20) the metric potentials are obtained as

$$A = A_1(1+z)^{\frac{-3n}{(n+2)}}, \quad \text{.................. (21)}$$

$$B = B_1(1+z)^{\frac{-3}{(n+2)}}. \quad \text{.................. (22)}$$

Where $A_1 = B_1^n$ and B_1 be the any arbitrary constant.

The Torsion scalar T becomes

$$T = -3H_0^2(1+(1+z)^{2+2\alpha}) \quad \text{.................. (23)}$$

The Energy density of the universe becomes

$$\rho = \frac{1}{16\pi}\left(\frac{3^{\beta}(-H_0^2(1+(1+z)^{2+2\alpha}))^{\beta}((2+n)^2-6(1+2n)\beta)}{(2+n)^2}\right). \quad \text{.................. (24)}$$

In our examined model, the energy density (ρ) exhibits a redshift-dependent behavior, where its value decreases as the redshift (z) increases. This phenomenon suggests that the energy density of the model is closely tied to the cosmic evolution, with its value diminishing as the universe undergoes expansion. The observed decrease in energy density with increasing redshift can be attributed to the dilution of energy due to the expanding universe. This behavior aligns with the fundamental principles of cosmology, which posit that the universe is homogeneous and isotropic on large scales. The implications of this behavior are far-reaching, particularly in the context of dark energy and cosmological models, as it provides valuable insights into the universe's evolution and the dynamics driving its expansion.

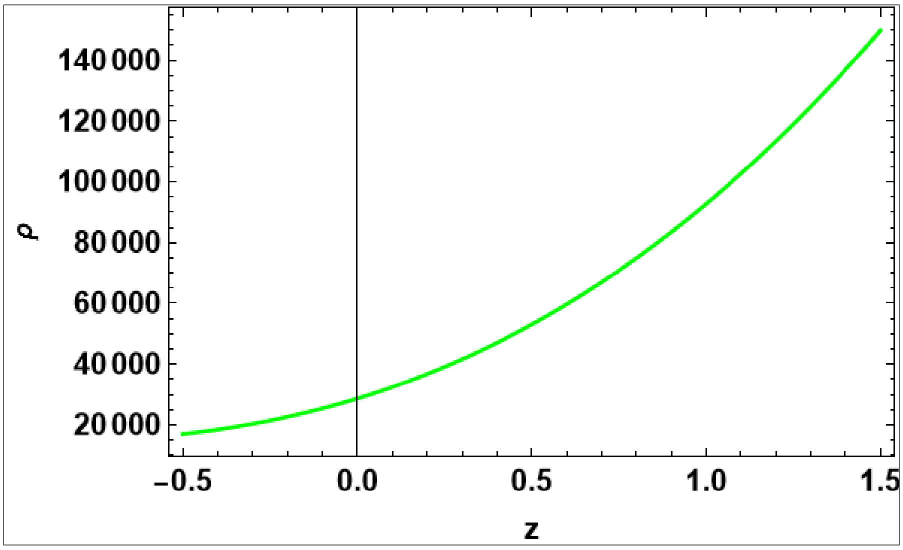

Figure 1 The behavior of energy density of the model versus redshift with the appropriate choice of constants

3.1. The pressure of the universe becomes

$$\bar{p} = -\frac{1}{16\pi}\left(\frac{(3^{\beta}(-H_0^2(1+(1+z)^{2+2\alpha}))^{\beta}(2+n-6\beta+(1+z)^{2+2\alpha}(2+n+2\beta(-5+2\alpha(-1+\beta)+2\beta))))}{(2+n+(2+n)(1+z)^{2+2\alpha})}\right) \text{..................} \quad (25)$$

In contrast, the isotropic pressure, as depicted in Fig. 2, displays a negative evolution, suggesting a potential driving force behind the universe's accelerated expansion. This observed behavior is consistent with the theoretical frameworks that govern cosmological evolution, providing a fascinating insight into the intricate dynamics of the universe.

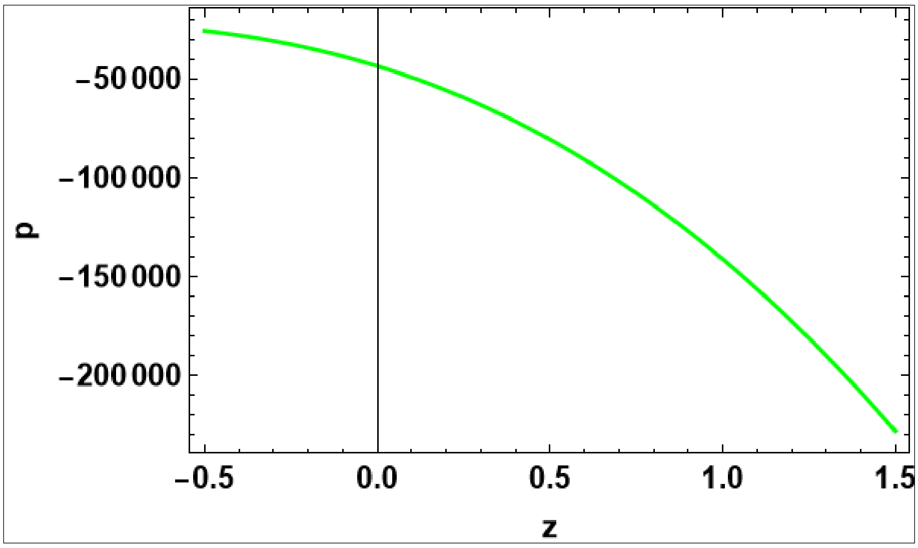

Figure 2 The behavior of isotropic pressure of the model versus redshift with the appropriate choice of constants

3.2. The Equation of state parameter of the universe becomes

$$\omega = \frac{(2+n)(-2-n+6\beta-(1+z)^{2+2\alpha}(2+n+2\beta(-5+2\alpha(-1+\beta)+2\beta)))}{4+4n+n^2-6\beta-12n\beta+(1+z)^{2+2\alpha}((2+n)^2-6(1+2n)\beta)}. \quad \text{..................} \quad (26)$$

Figure 3, ω, in the context of the current linear cosmological model. This parameter plays a pivotal role in cosmology, as it encapsulates the essential characteristics of the dominant component driving the universe's expansion. The EoS parameter's behavior is examined in relation to redshift (z), providing valuable insights into the model's cosmological implications. A notable feature of the figure is that the EoS parameter, ω, consistently remains below -1 across all redshift values. This phenomenon indicates that the model gravitates towards the phantom region, a hypothetical realm of dark energy thought to be responsible for the universe's accelerated expansion. The phantom region is distinguished by an EoS parameter less than -1 (ω < -1), signifying a dynamic and evolving dark energy component.

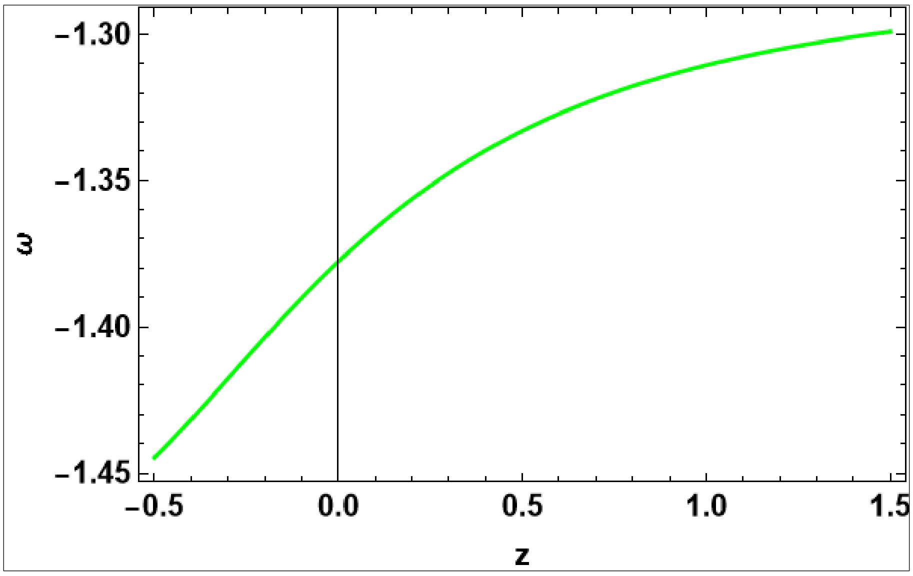

Figure 3 The behavior of equation of state parameter of the model versus redshift with the appropriate choice of constants

In this model, the EoS parameter is fixed at ω = -1.45, which corresponds to a static and unevolving dark energy component. However, the model's propensity towards the phantom region at high redshifts suggests that it exhibits a dynamic dark energy component. This behavior is congruent with observational evidence and offers a deeper understanding of the evolution of the universe's dark energy component.

4. Conclusion

In this paper, we investigated the Kantowski-Sachs cosmological model with perfect fluid within the framework of $f(T)$ gravity, specifically adopting the form $f(T) = T^{\beta}$. By employing the Hubble parameterization method and establishing a relationship between the metric potentials ($A = B^{n}$), we derived exact solutions to the field equations that describe an accelerating universe. The results provide valuable insights into the universe's evolution, particularly in the context of dark energy and cosmic acceleration.

We analysed key physical parameters, including the energy density, isotropic pressure, and equation of state (EoS) parameter, highlighting their redshift-dependent behaviour. The energy density was shown to decrease with increasing redshift, consistent with the expected cosmic expansion. The isotropic pressure exhibited a negative evolution, reinforcing its role in driving the acceleration of the universe. The EoS parameter consistently remained in the phantom regime ($\omega < -1$), suggesting that the model effectively describes a dynamic dark energy component.

Additionally, the torsion scalar's functional form and its impact on the model's dynamics were evaluated, emphasizing the flexibility of $f(T)$ gravity in addressing both early and late-time cosmological phenomena. Our findings align well with observational data, including Type Ia supernovae, CMB, and BAO measurements, demonstrating the model's compatibility with current cosmological benchmarks.

Overall, this study highlights the potential of $f(T)$ gravity in explaining the accelerated expansion of the universe and offers a robust framework for exploring alternative cosmological scenarios. Future work could extend this analysis to include more complex forms of $f(T)$, interactions with other fields, and additional observational constraints, further refining our understanding of the universe's evolution.

Compliance with ethical standards

The authors declare that the no conflict of interest..

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