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(Review Article)

A review of solvability of inequality-constrained optimization problems with non-linear functions

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Abstract

In this paper, we review techniques for solving inequality-constrained optimization problems with non-linear functions, emphasizing the Kuhn-Tucker (KT) method. The paper reviews their solvability ability as employed in solving inequality-constrained optimization problems. The necessary optimality conditions for obtaining optimal solutions for inequality-constrained problems are stated and discussed in this paper. The method of Kuhn-Tucker (KT) was used in solving for optimal solution of the inequality-constraints problem.

Keywords: Inequality Problem; Kuhn-Tucker; Effective Constraints; Complimentary Slackness; Necessary Conditions; Cardinality

1. Introduction

In general Optimization is a mathematical procedure for determining optimal allocation of scarce resources. In practice, most of the optimization problems encountered can be formulated as equality or inequality. It can either be a minimization problem or a maximization problem that may involve linear or nonlinear functions, and their approach to solving for optimal solutions are quite different. Many authors have applied varied methods to solve optimization problems of several types of optimization problems.

The paper in [1] used the method of Lagrange by converting inequality constraints into a new optimization problem with equality constraints and called the method a Valentine method for finite-dimensional optimization problems.

The authors in [2] developed a new wind energy project that requires studying many parameters to achieve maximum benefits at the cost of minimum environmental impacts. Using a Geographic Information System (GIS), they developed an analytical framework with fuzzy logic to evaluate the suitable site for turbines for optimum energy output.

Researchers in [3] considered a nonlinear constrained optimization model for selecting a pipe route with a minimum length that considers seabed topography, obstacles, and pipe curvature requirements.

In [4] and [5], the authors applied Newton Raphson's Iterative Algorithms to solve some constrained optimization problems. However, these methods are so cumbersome when there are multiple constraints with inequality constraint functions.

In [6], the authors developed a modified version of the Classical Lagrange Multiplier method and, applied it to solve convex quadratic optimization problems which they adapted from the first-order derivative test for optimality of the

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Lagrange function. In their method, they decompose the solution process into two independent ones, in which the primary and the secondary variables are solved independently before using the Lagrange multipliers method.

In our paper published recently, we discussed and reviewed the solvability analysis of equality-constrained problems with nonlinear functions using the Lagrange theorem to discuss the necessary optimality conditions [7].

This paper aims at using the theorem of Kuhn-Tucker (KT) to characterize the behavior of the objective function, A and the constraint function, h_i at local optima of inequality constrained optimization problems. The conditions which described the first –order necessary conditions for local optima in these problems. The KT approach to nonlinear programming generalizes the method of Lagrange multipliers.

1.1. Aim and Objectives

This paper aims to review the solvability of inequality-constrained optimization problems involving nonlinear functions and understand the conditions under which optimal solutions exist.

1.1.1. Objectives of the paper are

- Investigate the Necessary Conditions for Optimality – Establishing the fundamental conditions that must be satisfied for a solution to be optimal in inequality-constrained optimization problems involving nonlinear functions.
- Apply the Kuhn Tucker (KT) conditions – Demonstrating how the Kuhn Tucker conditions are used to identify extreme points of inequality constraints with nonlinear functions.

2. Methodology

The Method used in this paper is the Kuhn-Tucker method. The approach generalizes the method of Lagrange multipliers.

2.1. The Model for Inequality Constraints Optimization Problem

The model for Inequality constraints optimization problem is given by

Optimize $A(x)$

Subject to $x \in D \neq \emptyset$

Where $D = P \cap \{x \in \mathbb{R}^n : h_i(x) \geq 0, i = 1, \dots, l\}$,

$P \subset \mathbb{R}^n$ is open, and, $h_i : \mathbb{R}^n \to \mathbb{R}^l$, $A: \mathbb{R}^n \to \mathbb{R}$

2.2. Necessary Optimality Conditions for the Inequality Constraints Problem

The conditions described by KT theorem is viewed as first –order necessary conditions for local optima of nonlinear inequality constraints Problem.

Let $D = \{x \in \mathbb{R}^n : h(x) \geq 0\}$ be the inequality constraint set of an optimization problem and $h : \mathbb{R}^n \to \mathbb{R}^l$ be the constraint function, then h is called binding or effective constraint at a point $x^* \in D$ if $h(x^*) = 0$

Complementary slackness condition is the requirement that $\lambda_i \geq 0$ and $\lambda_i h_i(x) = 0$. The cardinality is the number of elements in the effective constraints, T and is denoted by $|T|$.

We say that a pair (x^*, λ^*) meets the first order necessary conditions for optimum point of the inequality constrained optimization problem if it satisfies $h(x^*) = 0$ as well as conditions $[KT-1]$and $[KT-2]$. The constraint qualification under inequality constraints is the condition in the theorem of Kuhn Tucker that the rank of the gradient of the effective constraints be equal to the cardinality of the effective constraints, $\rho \nabla h(x^*) = |T|$

The theorem of Kuhn and Tucker for inequality constrained optimization problems.

Theorem 1. Let $A : \mathbb{R}^n \to \mathbb{R}$ and $h_i : \mathbb{R}^n \to \mathbb{R}^l$ be c^1 functions, $i = 1, \dots, l$.

Suppose x^* is a local optimum of A on the set

$$D = P \cap \{x \in \mathbb{R}^n : h_i(x) \geq 0, i = 1, \dots, l\}, P \subset \mathbb{R}^n \text{ is open}.$$

Let $T \subset \{1,2,\dots,l\}$ denote the set of effective/binding constraints at x^* and let $h_T = h_{i(i \in T)}$

Suppose also that $\rho \nabla h(x^*) = |T|$, then there exists vector, $\lambda^* = (\lambda_1{}^*, \lambda_2{}^*, \dots, \lambda_l{}^*) \in \mathbb{R}^l$, such that the following conditions are satisfied,

$$[\boldsymbol{KT-1}]\ \lambda_i{}^* \geq 0, \text{ and, } \lambda_i{}^* h_i(x^*) = 0$$

$$[\boldsymbol{KT-2}]\ \nabla A(x^*) + \sum_{i=1}^{l} \lambda_i{}^* \nabla h_i(x^*) = 0.$$

Where $\lambda_i{}'s$ are called the Lagrangian multipliers associated with the local optimum x^*.

3. Applications

Here, we construct some examples and demonstrates how to verify the conditions of the theorem

Consider the problem:

Minimize $A(x) = -2x_1 - x_2$

Subject to $x_1 - x_2 \leq 0$

$x_1{}^2 + x_2{}^2 \leq 4$

$x_1, x_2 \geq 0$

$h_1(x) = x_1 - x_2 \leq 0,$

$h_2(x) = x_1{}^2 + x_2{}^2 \leq 4.$

3.1. Solution

The problem is a non-linear programming problem since the second constraints is nonlinear.

$$A(x) = -2x_1 - x_2$$

$$h_1(x) = x_1 - x_2$$

$$h_2(x) = x_1{}^2 + x_2{}^2 - 4$$

$$\nabla A(x) = \begin{bmatrix} \frac{\partial A}{\partial x_1} & \frac{\partial A}{\partial x_2} \end{bmatrix} = [-2 \quad -1]$$

$$\nabla h_1(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \end{bmatrix} = [1 \quad -1]$$

$$\nabla h_2(x) = \begin{bmatrix} \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} = [2x_1 \quad 2x_2]$$

Verifying the KT conditions;

$$\nabla A(x) + \sum_{i=1}^{l} \lambda_i \nabla h_i(x) = 0.$$

$$\Rightarrow [-2 \quad -1] + \lambda_1[1 \quad -1] + \lambda_2[2x_1 \quad 2x_2] = 0$$

$$-2 + \lambda_1 + 2\lambda_2 x_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$-1 - \lambda_1 + 2\lambda_2 x_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\lambda_i h_i(x) = 0, \forall\, i = 1,2.$$

$$\Rightarrow \lambda_1(x_1 - x_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\lambda_2({x_1}^2 + {x_2}^2 - 4) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$h_i(x) \leq 0, \forall i = 1,2.$$

$$\Rightarrow x_1 - x_2 \leq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$ {x_1}^2 + {x_2}^2 - 4 \leq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$\lambda_i \geq 0, \forall i = 1,2.$$

$$\Rightarrow \lambda_1 \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\lambda_2 \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

3.1.1. Solving simultaneously,

From (3); $\lambda_1(x_1 - x_2) = 0$

Either $\lambda_1 = 0$ *or* $x_1 - x_2 = 0$

From (4); $\lambda_2({x_1}^2 + {x_2}^2 - 4) = 0$

$$\text{Either } \lambda_2 = 0 \text{ or } {x_1}^2 + {x_2}^2 - 4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

From the above, we can have the following cases;

Case1: $\lambda_1 = 0$; $\lambda_2 = 0$

Substitute, $\lambda_1 = 0$ *and* $\lambda_2 = 0$ in (1)*and* (2);

$-2 = 0$; $-1 = 0$ … *does not hold*

Case 2: $\lambda_1 = 0$; $\lambda_2 \neq 0$

Substitute $\lambda_1 = 0$ *and* $\lambda_2 \neq 0$ in (1)*and* (2);

$$-2 + 2\lambda_2 x_1 = 0 \Rightarrow x_1 = \frac{1}{\lambda_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$-1 + 2\lambda_2 x_2 = 0 \Rightarrow x_2 = \frac{1}{2\lambda_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Also, from (9), when $\lambda_2 \neq 0, {x_1}^2 + {x_2}^2 - 4 = 0$

Substitute $x_1 = \frac{1}{\lambda_2}$ *and* $x_2 = \frac{1}{2\lambda_2}$ *in* ${x_1}^2 + {x_2}^2 - 4 = 0$

$$\Rightarrow \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{2\lambda_2}\right)^2 = 4$$

$$\frac{1}{{\lambda_2}^2} + \frac{1}{4{\lambda_2}^2} = 4$$

$$\frac{4+1}{4{\lambda_2}^2} = 4$$

$$\frac{5}{4{\lambda_2}^2} = 4$$

$$5 = 16{\lambda_2}^2$$

$${\lambda_2}^2 = \frac{5}{16}$$

$$\lambda_2 = \pm\sqrt{\frac{5}{16}}$$

Since λ_2 cannot be negative, $\lambda_2 = +\sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$

Substitute $\lambda_2 = \frac{\sqrt{5}}{4}$ $in\ (10)$; $x_1 = \frac{1}{\lambda_2} = \frac{4}{\sqrt{5}}$

Substitute $\lambda_2 = \frac{\sqrt{5}}{4}$ $in\ (11)$; $x_2 = \frac{1}{2\lambda_2} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$

Since $x_1 = \frac{4}{\sqrt{5}}$ and $x_2 = \frac{2}{\sqrt{5}}$ satisfy all the necessary conditions above, then $x^* = (x_1, x_2) = \left(\frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ is the optimum solution.

3.1.2. Hence, theorem 1 is verified in this example.

Consider the problem where the objective function and the constraint sets are non-linear.

$Minmize\ A(x) = 6(x_1 - 10)^2 + 4(x_2^2 - 12.5)^2$

$subject\ to$:

$$x_1^2 + (x_2 - 5)^2 \leq 50,$$

$$x_1^2 + 3x_2 \leq 250$$

$$(x_1 - 6)^2 + x_2^2 \leq 37.$$

$$A: \mathbb{R}^2 \to \mathbb{R},\ h: \mathbb{R}^2 \to \mathbb{R}^3.$$

Were, $h = h_1, h_2, h_3$

$$h_1 = x_1^2 + (x_2 - 5)^2 - 50 \leq 0$$

$$h_2 = x_1^2 + 3x_2^2 - 250 \leq 0.$$

$$h_3 = (x_1 - 6)^2 + x_2^2 - 37 \leq 0.$$

$$\nabla A(x) = \begin{bmatrix} \frac{\partial A}{\partial x_1} & \frac{\partial A}{\partial x_2} \end{bmatrix} = [12(x_1 - 10) \quad 8(x_2 - 12.5)]^T$$

$$\nabla h_1(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \end{bmatrix} = [2x_1 \quad 2(x_2 - 5)]^T$$

$$\nabla h_2(x) = \begin{bmatrix} \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} = [2x_1 \quad 6x_2]^T$$

$$\nabla h_3(x) = \begin{bmatrix} \frac{\partial h_3}{\partial x_1} & \frac{\partial h_3}{\partial x_2} \end{bmatrix} = [2(x_1 - 6) \quad 2x_2]^T$$

Verifying the condition

$$\nabla A(x) + \sum_{i=1}^{l} \lambda_i \nabla h_i\,(x) = 0'$$

$$\Rightarrow \begin{bmatrix} 12(x_1 - 10) \\ 8(x_2 - 12.5 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2x_1 \\ 2(x_2 - 5) \end{bmatrix} + \lambda_2 \begin{bmatrix} 2x_1 \\ 6x_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} 2(x_1 - 6) \\ 2x_2 \end{bmatrix} = 0$$

$$12(x_1 - 10) + 2x_1\lambda_1 + 2x_1\lambda_{2+}2(x_1 - 6)\lambda_3 = 0. \quad \text{………………}(12)$$

$$8(x_2 - 12.5) + 2(x_2 - 5)\lambda_1 + 6x_2\lambda_{2+}2x_2\lambda_3 = 0. \quad \text{………………}(13)$$

Solving equations (12) and (13) simultaneously for $(x_{1,}x_2, \lambda_1, \lambda_2, \lambda_3)$ gives

$$x_{1,} = 7, x_2 = 6, \lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 4,$$

$$(\lambda_1^*, (\lambda_2^*, \lambda_3^*) = (2{,}0{,}4).$$

Therefore, $(x_{1,}^* x_2^*) = (7{,}6)$

Now, we test if $(x_{1,}^* x_2^*)$is a candidate for optimal solution to the problem. First, we verify the set of effective constraint denoted by T by testing the feasibility of $(x_{1,}^* x_2^*)$

$$h_1(7{,}6) = 0 \le 0$$

$$h_2(7{,}6) = -93 \le 0$$

$$h_3(7{,}6) = 0 \le 0$$

Therefore, the effective constraints $T = \{1{,}3\}$

We check the rank of the gradient of the effective constraints, $\rho\nabla h_T\left((x_{1,}^* x_2^*\right)$

$$\nabla h_1(7{,}6_,) = \begin{bmatrix} 14 \\ 2 \end{bmatrix}$$

$$\nabla h_3(7{,}6_,) = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$\nabla A(7{,}6_,) = \begin{bmatrix} -36 \\ -52 \end{bmatrix}$$

$$\nabla h_T(7{,}6_,) = \begin{bmatrix} 14 & 2 \\ 2 & 12 \end{bmatrix}$$

Since the determinant of $\nabla h_T(7{,}6_,) \ne 0 \rightarrow \rho\nabla h_T(7{,}6) = 2 = T$

Where $|T|$ denotes the cardinality of T,that is, the number of elements in the set, T. Then, there exists a vector $\lambda = (\lambda_1^*, (\lambda_2^*, \lambda_3^*) \ge 0$ such that $\nabla A(x) + \sum_{i\in T}^{2} \nabla\, h_i(x^*)$=0

$$\begin{bmatrix}-36\\-52\end{bmatrix} + \lambda_1^* \begin{bmatrix}14\\2\end{bmatrix} + \lambda_3^* \begin{bmatrix}2\\12\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Solving equation (14), we have,

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in \mathbb{R}^3 = (2,0,4).$$

Hence, $(x_{1,}^* x_2^*) = (7,6)$ satisfies the Kuhn-Tucker condition and therefore $(x_{1,}^* x_2^*) = (7,6)$ is a candidate for an optimal solution of the problem given.

4. Conclusion

Here in, we have been able to discuss the theorem of Kuhn Tucker (KT) and its application in obtaining the optimal solutions of inequality-constrained problems with nonlinear functions, verifying all the conditions of KT is an interesting part of the steps. This result shows that the Kuhn-Tucker method is very efficient in obtaining the optimal solutions of inequality-constrained problems.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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